



# Groupoids and strong shape

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Received 31 January 2003; accepted 6 October 2003

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## Abstract

Ordinary shape equivalences can be defined in full generality, for every pair  $(\mathcal{C}, \mathcal{K})$  of abstract categories. In this paper we consider pairs  $(\mathcal{C}, \mathcal{K})$  enriched over the category of groupoids and show how it is possible, in such a setting, to introduce a general concept of strong shape equivalence. A category  $\Sigma(\mathcal{C}, \mathcal{K})$ , having the same objects as  $\mathcal{C}$ , and a functor  $\sigma : \mathcal{C} \rightarrow \Sigma(\mathcal{C}, \mathcal{K})$  are defined, so that  $\sigma$  inverts exactly the class of strong shape equivalences.

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*MSC:* 55P55; 55P65; 18B30; 18B40; 55U99; 54B30

*Keywords:* Full image; Proreflector; Component functor; Groupoid enriched category; Track groupoid; Strong shape equivalence

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## 0. Introduction

Soon after K. Borsuk founded Shape Theory in 1968, it was observed by W. Holsztyński [16] that it could be formulated in an abstract categorical setting. S. Mardešić [21] extended the notion of shape to arbitrary topological spaces, also recognizing the essential categorical features of the theory, adopting what is now called the inverse system approach. Later on, Le Van (see [10]) introduced the notion of shape for full embeddings  $\mathcal{K} \subset \mathcal{C}$  of abstract categories and Deleanu and Hilton [10] gave a categorical notion of shape for arbitrary functors  $E : \mathcal{K} \rightarrow \mathcal{C}$ . See also the work of A. Frei [13], showing the many implications of this categorical point of view. It could be said that categorical shape is an autonomous

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branch of Category Theory, besides having greatly contributed to a better and deeper understanding of Borsuk's and Mardešić's shape. A neat and comprehensive treatment of categorical shape theory is the book by Cordier and Porter [8].

Since a stronger version of topological shape was introduced from about 1975 (e.g., [7,12,22]), also the need for a categorical strong shape theory, richer in structure than the previous one, has appeared, which could capture the geometrical complexity of the matter [20]. In this direction one should mention the remarkable paper by Batanin [3], where a complete categorical description of the category of coherent prohomotopy theory [9,20] is given, together with a satisfactory categorical setting for strong shape theory, also with applications to several other areas.

In two recent papers [25,26] we have studied in some detail shape and strong shape equivalences. The concept of shape equivalence is a very general one and it can be defined for an arbitrary functor  $E: \mathbf{K} \rightarrow \mathbf{C}$ . A morphism  $f \in \mathbf{C}(X, Y)$  is a shape equivalence with respect to  $E$  whenever the induced transformation  $f^*: \mathbf{C}(Y, E(-)) \rightarrow \mathbf{C}(X, E(-))$  is a natural isomorphism. Of particular interest is the case where  $E$  is the embedding of a full subcategory  $\mathbf{K}$  of  $\mathbf{C}$ . As for strong shape equivalences, it turns out that a continuous map  $f: X \rightarrow Y$  is such if and only if it induces a natural family of equivalences of track groupoids [5]  $f_P^\# : \pi P^Y \rightarrow \pi P^X$ , for all absolute neighborhood retracts  $P$ . We consider the 2-category  $\mathcal{TOP}$  of topological spaces, with its enrichment over the category  $\mathbf{GPD}$  of groupoids. This suggests that one could define the concept of strong shape equivalence in the more abstract setting of a pair  $(\mathcal{C}, \mathcal{K})$  of g.e. categories (that is, enriched over  $\mathbf{GPD}$ ) or even for a 2-functor  $F: \mathcal{K} \rightarrow \mathcal{C}$ , almost in the same spirit of what happened for ordinary shape equivalences, without considering here coherence conditions. This point of view leads us to the definition of a category  $\Sigma(\mathcal{C}, \mathcal{K})$  and of a functor  $\sigma: \mathcal{C} \rightarrow \Sigma(\mathcal{C}, \mathcal{K})$  which inverts exactly the strong shape equivalences of  $\mathcal{C}$  with respect to  $\mathcal{K}$ .

Every g.e. category  $\mathcal{C}$  has a germ of homotopy in its own structure that becomes evident when one applies the *component* functor  $\Delta: \mathbf{GPD} \rightarrow \mathbf{SET}$ . This fact allows one to define, for such a g.e. category  $\mathcal{C}$ , besides a homotopy category  $h\mathcal{C}$  also a shape category  $\mathcal{S}(\mathcal{C}, \mathcal{K})$ , for every chosen full subcategory  $\mathcal{K}$  of  $\mathcal{C}$ . In the case  $\mathcal{C} = \mathcal{TOP}$  and  $\mathcal{K} = \mathcal{ANR}$ , the g.e. category of absolute neighborhood retracts, it is immediate to see that  $h\mathbf{TOP}$  is the usual homotopy category of all topological spaces and homotopy classes of continuous maps, while  $\mathcal{S}(\mathcal{TOP}, \mathcal{ANR})$  is isomorphic to the ordinary shape category for the pair  $(h\mathbf{TOP}, h\mathbf{ANR})$ . We use here a different notation for, e.g., the 2-category  $\mathcal{TOP}$  and its underlying category  $\mathbf{TOP}$ .

In the first part of the paper we consider the full image factorization of a functor [6,23], since almost all relevant categories and functors that appear in shape and strong shape are closely related to such a construction. Moreover, it allows one also to clarify (Proposition 1.2) in a simple way the link between the two approaches to ordinary shape by natural transformations and by morphisms of inverse systems (see also [8]). The second part is devoted to g.e. categories and the explanation of what we call the component functor. Here we give our main definitions. The last section is devoted to shape and strong shape equivalences. Also we consider the relationship between the Lisica–Mardešić strong shape category  $SSh(\mathbf{TOP})$  of topological spaces and the category  $\Sigma(\mathcal{TOP}, \mathcal{ANR})$ . The author is grateful to the referee for addressing his attention to the work of Batanin [2,3] and also

for pointing out that the approach in this paper concerning the construction of the category  $\Sigma(\mathcal{C}, \mathcal{K})$  can be fitted into the theory developed in [3].

## 1. Full image and shape theory

Throughout the paper we use an elementary categorical notion—the *full image factorization* of a functor  $F : A \rightarrow B$ —which reveals to be a useful tool in many situations. Let  $A_F$  be the category having the same objects as  $A$  and the morphisms of  $B$ , in the sense that  $A_F(X, X')$  is identified with  $B(F(X), F(X'))$ . There are functors  $F_0 : A \rightarrow A_F$  and  $F_1 : A_F \rightarrow B$ , defined by  $F_0(X) = X$  and  $F_0(a) = F(a)$ , for  $a : X \rightarrow X'$  in  $A$ , and  $F_1(X) = F(X)$ ,  $F_1(u) = u$ , for  $u : F(X) \rightarrow F(X')$  in  $B$ . They give a factorization  $F = F_1 \circ F_0$  of  $F$  which is uniquely determined, up to isomorphisms, among all factorizations  $F = H'' \circ H'$ , where  $H'$  is bijective on objects and  $H''$  is fully faithful [23, 21.2]. The following properties are easy to prove and will be useful in the sequel.

**Lemma 1.1.** *Let  $F : A \rightarrow B$  and  $G : B \rightarrow C$  be functors.*

- (i) *There is a unique functor  $T : A_F \rightarrow A_{G \circ F}$  such that  $T \circ F_0 = (G \circ F)_0$  and  $(G \circ F)_1 \circ T = G \circ F_1$ . If  $G$  is fully faithful, then  $T$  is an isomorphism.*
- (ii) *There is a unique functor  $V : A_{G \circ F} \rightarrow B_G$  such that  $V \circ (G \circ F)_0 = G_0 \circ F$  and  $G_1 \circ V = (G \circ F)_1$ . If  $F$  is bijective on objects, then  $V$  is an isomorphism.*

In the following, given a category  $K$ , we shall denote by  $\{K, \text{SET}\}$  the category (possibly in a larger universe) of functors from  $K$  to the category  $\text{SET}$  of sets, and the natural transformations between them.

Let  $(C, K)$  be a pair of categories with  $E : K \rightarrow C$  the inclusion functor. The shape category of the functor  $E$  or, of the pair  $(C, K)$ , has a nice categorical description. The usual Yoneda embedding  $Y_K : K \rightarrow \{K, \text{SET}\}^0$  has an extension

$$\gamma : C \rightarrow \{K, \text{SET}\}^0,$$

defined, for every  $X \in C$ , by  $\gamma(X) = C(X, E(-)) : K \rightarrow \text{SET}$ .

The *shape category*  $Sh(C, K)$  of the pair  $(C, K)$  is the full image of  $\gamma$ :

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & \{K, \text{SET}\}^0 \\ & \searrow \gamma_0 & \nearrow \gamma_1 \\ & Sh(C, K) & \end{array}$$

Hence, its objects are those of  $C$  and the morphisms can be described by

$$Sh(X, Y) \cong \text{Nat}(C(Y, E(-)), C(X, E(-))),$$

where  $\text{Nat}$  means class of natural transformations. The shape functor  $S = \gamma_0$  is the identity on objects and sends every map  $f : X \rightarrow Y$  to the natural transformation  $f^* : C(Y, E(-)) \rightarrow C(X, E(-))$  given by composition with  $f$ . Such a description can

be generalized to the case of an arbitrary functor  $E : K \rightarrow C$ , see [10,13], and finds a number of applications in general category theory. A more geometrical approach to shape was settled in [19,21], using inverse systems.

Let  $\text{pro}K$  denote the category of inverse systems in  $K$  (see [21] for all details). An object in  $\text{pro}K$  will be written  $X = (X_i)_{i \in I}$ , where each  $X_i$  is in  $K$  and  $I$  is the directed index set.  $K$  is said to be proreflective in  $C$  if there is a functor  $P : C \rightarrow \text{pro}K$ , called the *proreflector*, having the property that, for all  $X \in C$ , there is a morphism  $X \rightarrow P(X)$  in  $\text{pro}C$  which is initial with respect to all morphisms  $X \rightarrow K$  in  $\text{pro}C$ , with  $K \in \text{pro}K$ . The Grothendieck functor

$$L : \text{pro}K \rightarrow \{K, \text{SET}\}^0, \quad L(X) = \varinjlim_i K(X_i, -),$$

also extends the Yoneda embedding  $Y_K$  and is fully faithful.

Notice that, if  $K$  is proreflective in  $C$  and  $P(X) = (X_i)_{i \in I}$ , then there is a natural isomorphism

$$C(X, E(-)) \cong \varinjlim_i K(X_i, -),$$

see, for instance, [8, p. 53], hence  $\gamma = L \circ P$  holds.

The link with the inverse system approach to shape is given by the following

**Proposition 1.2.** *Let  $K$  be proreflective in  $C$  with proreflector  $P : C \rightarrow \text{pro}K$ . The shape category of the pair  $(C, K)$  is the full image of the functor  $P$ .*

**Proof.** The assertion follows from Lemma 1.1(i), since  $L$  is fully faithful. Note that  $L \circ P_1 = \gamma_1$ .  $\square$

The above applies in particular to the case where  $C = h\text{TOP}$  is the homotopy category of topological spaces and  $K = h\text{ANR}$  is its full subcategory of spaces having the homotopy type of ANRs. The proreflector  $P : h\text{TOP} \rightarrow \text{pro}(h\text{ANR})$  is that assigning to every space its Čech system [21]. Notice that the construction above is not sensible of any notion of homotopy, unless one puts it directly considering homotopy categories, e.g., of spaces, as starting point. In the sequel we shall improve the above introducing a germ of homotopy just in the definitions.

## 2. g.e. categories and shape

Recall that a groupoid is a small category whose morphisms are all invertible.  $\text{GPD}$  will denote the category of groupoids and functors between them. It is a complete and cocomplete category [15].

If  $G \in \text{GPD}$  and  $x \in G$ , the *component* of  $x$ , written  $[x]$ , is the equivalence class of  $x$  under the relation

$$x \sim y \Leftrightarrow G(x, y) \neq \emptyset.$$

Let us denote by  $\Delta(G)$  the set of components of the groupoid  $G$ . Every functor of groupoids  $f: G \rightarrow H$  induces a function  $\Delta(f): \Delta(G) \rightarrow \Delta(H)$  given by  $\Delta(f)([x]) = [f(x)]$ . The data above define the *component functor* [18, p. 295]

$$\Delta: \mathbf{GPD} \rightarrow \mathbf{SET}, \quad G \mapsto \Delta(G), \quad f \mapsto \Delta(f).$$

Let, moreover,  $D: \mathbf{SET} \rightarrow \mathbf{GPD}$  be the *discrete* functor. It takes a set  $S$  to the discrete category  $D(S)$  over  $S$ . Then  $\Delta \circ D = 1_{\mathbf{SET}}$ .

**Lemma 2.1.** *The component functor  $\Delta$  is left adjoint to the discrete functor  $D$ . In particular,  $\Delta$  commutes with colimits.*

**Definition 2.2.** A groupoid enriched category (g.e. category, for short)  $\mathcal{C}$  is a 2-category [4, 17] whose 2-morphisms are all invertible. As a matter of notations, we will write  $\pi Y^X$  for the groupoid with objects the 1-morphisms  $X \rightarrow Y$  of  $\mathcal{C}$  and morphisms the 2-morphisms  $f \Rightarrow g$  of  $\mathcal{C}$  between them. Moreover,  $[X, Y]$  will denote the set  $\Delta(\pi Y^X)$  of its components. Composition of 1-morphisms is usually denoted as  $g \circ f$ , while composition of 2-morphisms and mixed composition is denoted by  $\alpha * \beta$  and  $\alpha * f$ .

If  $\mathcal{C}$  is a g.e. category, then  $\mathbf{C}$  will denote its underlying category, formed by the objects and the 1-morphisms of  $\mathcal{C}$ .

A g.e. category  $\mathcal{C}$  is said to be locally discrete if  $\pi Y^X$  is discrete, for all  $X, Y \in \mathcal{C}$ . In such a case  $\mathcal{C}$  can be identified with the underlying category  $\mathbf{C}$ , so we shall write simply  $\mathbf{C}$  for it.

Every g.e. category  $\mathcal{C}$  has a *homotopy category*  $h\mathbf{C}$ . This is the ordinary category having the same objects of  $\mathcal{C}$  and morphisms defined by

$$h\mathbf{C}(X, Y) = [X, Y].$$

As for the composition, observe that the component functor commutes with products [14].

**Examples 2.3.** (1)  $\mathcal{GPD}$  is the g.e. category whose underlying category is  $\mathbf{GPD}$  and where a 2-morphism  $\alpha: f \Rightarrow g$  is a natural transformation of functors of groupoids. Notice that  $\alpha$  has to be a natural isomorphism. It is also called a homotopy connecting  $f$  and  $g$ .

(2)  $\mathcal{TOP}$  is the g.e. category whose underlying category is the category  $\mathbf{TOP}$  of topological spaces and continuous maps, and where a 2-morphism  $\alpha: f \Rightarrow g$ , also called a *track* from  $f$  to  $g$ , is a relative homotopy class of homotopies connecting  $f$  and  $g$ . In such a case  $\pi Y^X$  is the so-called *track groupoid* of  $Y$  under  $X$  [18, 5]. It is plain that  $[X, Y]$  is here the set of homotopy classes of maps from  $X$  to  $Y$ , so that  $h\mathbf{TOP}$  is the usual homotopy category of spaces.

(3) Every ordinary category can be viewed as a g.e. category where all 2-morphisms are identities.

(4) For every g.e. category  $\mathcal{C}$  one can construct the g.e. category  $\mathbf{pro}\mathcal{C}$  in such a way that the underlying category  $\mathbf{pro}\mathcal{C}$  is the category of inverse systems in  $\mathcal{C}$ . In fact, for  $X = (X_i)_{i \in I}$  and  $Y = (Y_j)_{j \in J}$ , one can define  $\pi Y^X = \lim_{\leftarrow j} \lim_{\rightarrow i} \pi Y_j^{X_i}$ .  $h(\mathbf{pro}\mathbf{TOP})$  is the naive homotopy category of  $\mathbf{pro}\mathbf{TOP}$  [20]. In the next section we shall give some more insights on the g.e. structure of  $\mathbf{pro}\mathcal{C}$ .

Let  $\mathcal{C}, \mathcal{D}$  be g.e. categories. The following definitions can be found, e.g., in [4]:

- a g.e. functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of an object function which sends every object  $X \in \mathcal{C}$  to an object  $F(X) \in \mathcal{D}$  and, for every  $X, Y \in \mathcal{C}$ , of functors  $F_{X,Y} : \pi Y^X \rightarrow \pi F(Y)^{F(X)}$ , compatible with identities and composition,
- a g.e. natural transformation  $t : F \rightarrow G$  is a family  $t = \{t_X : F(X) \rightarrow G(X) \mid X \in \mathcal{C}\}$  of 1-morphisms of  $\mathcal{D}$  having the property that  $\mathcal{D}(1, t_Y) \circ F_{X,Y} = \mathcal{D}(t_X, 1) \circ G_{X,Y}$ , as functors of groupoids. In diagram

$$\begin{array}{ccc} \pi Y^X & \xrightarrow{F_{X,Y}} & \pi F(Y)^{F(X)} \\ G_{X,Y} \downarrow & & \downarrow \mathcal{D}(1, t_Y) \\ \pi G(Y)^{(G)X} & \xrightarrow{\mathcal{D}(t_X, 1)} & \pi G(Y)^{F(X)} \end{array}$$

- if  $t, t' : F \rightarrow G$  are two g.e. natural transformations, a modification  $\alpha : t \Rightarrow t'$  consists of a family  $\alpha = \{\alpha_X : t_X \Rightarrow t'_X \mid X \in \mathcal{C}\}$  of 2-morphisms of  $\mathcal{D}$ , such that  $\mathcal{D}(1, \alpha_Y) \circ F_{X,Y} = \mathcal{D}(\alpha_X, 1) \circ G_{X,Y}$ , again as morphisms of groupoids. In diagram

$$\begin{array}{ccc} \pi Y^X & \xrightarrow{F_{X,Y}} & \pi F(Y)^{F(X)} \\ G_{X,Y} \downarrow & & \downarrow \mathcal{D}(1, \alpha_Y) \\ \pi G(Y)^{(G)X} & \xrightarrow{\mathcal{D}(\alpha_X, 1)} & \pi G(Y)^{F(X)} \end{array}$$

The data above allow one to define (possibly in a larger universe) the g.e. category  $[[\mathcal{C}, \mathcal{D}]]$  with objects the g.e. functors, 1-morphisms the g.e. natural transformations and 2-morphisms the modifications.

**Remark 2.4.** (1) A g.e. functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , where  $\mathcal{D}$  is a locally discrete g.e. category, is an ordinary functor on the underlying categories, which is constant on each component. A g.e. natural transformation between such functors is also an ordinary natural transformation and there are no modifications apart from identities.

(2) For a g.e. category  $\mathcal{C}$ , its homotopy functor  $h_{\mathcal{C}} : \mathcal{C} \rightarrow h\mathcal{C}$  is the g.e. functor which is the identity on objects, sends each 1-morphism  $f \in \pi Y^X$  to its component  $[f] \in [X, Y]$  and each 2-morphism to an identity.

(3) The full image construction of Section 1 extends easily to a g.e. functor  $F : \mathcal{A} \rightarrow \mathcal{B}$ . In such a case  $\mathcal{A}_F(X, X') = \pi X'^X$  is identified with the groupoid  $\mathcal{B}(F(X), F(X')) = \pi F(X')^{F(X)}$  and the related version of Lemma 1.1 continues to hold.

There is a g.e. version of the Yoneda embedding for a g.e. category  $\mathcal{C}$ , denoted by

$$\mathcal{Y}_{\mathcal{C}} : \mathcal{C} \rightarrow [[\mathcal{C}, \mathcal{GPD}]]^0,$$

which is described as follows:

- an object  $X \in \mathcal{C}$  is sent to the g.e. functor  $\pi(-)^X : \mathcal{C} \rightarrow \mathcal{GPD}$ , which takes a  $P \in \mathcal{C}$  to the groupoid  $\pi P^X$ ,

- a 1-morphism  $f: X \rightarrow Y$  of  $\mathcal{C}$  is sent to the g.e. natural transformation  $f^\#: \pi(-)^Y \rightarrow \pi(-)^X$ , induced by  $f$ . Notice that  $f^\# = \{f_P^\#: \pi P^Y \rightarrow \pi P^X \mid P \in \mathcal{C}\}$  has to be a natural family of morphisms of groupoids where, for each  $u \in \pi P^Y$ ,  $f_P^\#(u) = u \circ f$ , and, for  $t: u \Rightarrow v$  in  $\pi P^Y$ ,  $f_P^\#(t) = t * f: u \circ f \Rightarrow v \circ f$ ,
- a 2-morphism  $\alpha: f \Rightarrow g$  of  $\mathcal{C}$  is sent to the modification  $\alpha^\#: f^\# \Rightarrow g^\#$  where, for every  $P \in \mathcal{C}$  and  $u \in \pi P^Y$ ,  $\alpha_P^\#(u) = u * \alpha: u \circ f \Rightarrow u \circ g$ .

**Remark 2.5.** (1) Every g.e. functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  gives a functor  $hF: h\mathcal{C} \rightarrow h\mathcal{D}$ , defined by  $hF(X) = F(X)$ , for  $X \in \mathcal{C}$ , and  $hF([f]) = [F(f)]$ , for  $f$  a 1-morphism of  $\mathcal{C}$ .

(2) The homotopy category  $h\mathcal{C}$  of a g.e. category  $\mathcal{C}$  can be obtained as the full image of the g.e. functor  $\Delta^* \circ \mathcal{Y}_{\mathcal{C}}$ , as described in the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{Y}_{\mathcal{C}}} & \llbracket \mathcal{C}, \mathcal{GPD} \rrbracket^0 \xrightarrow{\Delta^*} \llbracket \mathcal{C}, \text{SET} \rrbracket^0 \\ & \searrow h_{\mathcal{C}} = (\Delta^* \circ \mathcal{Y}_{\mathcal{C}})_0 & \nearrow (\Delta^* \circ \mathcal{Y}_{\mathcal{C}})_1 \\ & h\mathcal{C} & \end{array}$$

where  $\Delta^*$  denotes composition with the component functor, which is considered here as g.e. functor, in the obvious way.

(3) Given a g.e. pair  $(\mathcal{C}, \mathcal{K})$ , with  $E: \mathcal{K} \rightarrow \mathcal{C}$  the inclusion, the g.e. Yoneda embedding  $\mathcal{Y}_{\mathcal{K}}$  has the extension

$$\Gamma_E: \mathcal{C} \rightarrow \llbracket \mathcal{K}, \mathcal{GPD} \rrbracket^0$$

defined by  $\Gamma_E(X) = \pi E(-)^X$ ,  $\Gamma_E(f) = f^\#$  and  $\Gamma_E(\alpha) = \alpha^\#$ .

(4) If  $h_{\mathcal{GPD}}$  is the homotopy functor for  $\mathcal{GPD}$ , the composition

$$\mathcal{C} \xrightarrow{\Gamma_E} \llbracket \mathcal{K}, \mathcal{GPD} \rrbracket^0 \xrightarrow{h_{\mathcal{GPD}}^*} \llbracket \mathcal{K}, h\mathcal{GPD} \rrbracket^0$$

takes an object  $X \in \mathcal{C}$  to the functor  $h_{\mathcal{GPD}} \circ \pi(-)^X: \mathcal{K} \rightarrow h\mathcal{GPD}$  and a 1-morphism  $f: X \rightarrow Y$  to the natural transformation  $h_{\mathcal{GPD}} \circ f^\# = \{[f_P^\#]: \pi P^Y \rightarrow \pi P^X \mid P \in \mathcal{K}\}$ . It follows that, if  $g: G \rightarrow H$  is in the same component of  $f$ , then  $h_{\mathcal{GPD}} \circ f^\# = h_{\mathcal{GPD}} \circ g^\#$ . Hence the functor  $h_{\mathcal{GPD}}^* \circ \Gamma_E$  has a lifting to the homotopy category of  $\mathcal{C}$ , written  $\Gamma_E^h: h\mathcal{C} \rightarrow \llbracket \mathcal{K}, h\mathcal{GPD} \rrbracket^0$ , which commutes the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Gamma_E} & \llbracket \mathcal{K}, \mathcal{GPD} \rrbracket^0 \\ h_{\mathcal{C}} \downarrow & & \downarrow h_{\mathcal{GPD}}^* \\ h\mathcal{C} & \xrightarrow{\Gamma_E^h} & \llbracket \mathcal{K}, h\mathcal{GPD} \rrbracket^0 \end{array}$$

We are led to give the following definitions.

**Definition 2.6.** The shape category  $\mathcal{S}(\mathcal{C}, \mathcal{K})$  of the g.e. pair  $(\mathcal{C}, \mathcal{K})$  is the full image of the g.e. functor  $\Delta^* \circ \Gamma_E : \mathcal{C} \rightarrow \llbracket \mathcal{K}, \mathbf{SET} \rrbracket^0$ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Gamma_E} & \llbracket \mathcal{K}, \mathcal{GPD} \rrbracket^0 \xrightarrow{\Delta^*} \llbracket \mathcal{K}, \mathbf{SET} \rrbracket^0 \\ & \searrow S=(\Delta^* \circ \Gamma_E)_0 & \nearrow (\Delta^* \circ \Gamma_E)_1 \\ & \mathcal{S}(\mathcal{C}, \mathcal{K}) & \end{array}$$

$\mathcal{S}(\mathcal{C}, \mathcal{K})$  is locally discrete, hence it can be identified with its underlying category. Since  $\Delta = h\Delta \circ h\mathcal{GPD}$ , it follows from Lemma 1.1(i) that  $\mathcal{S}(\mathcal{C}, \mathcal{K})$  is isomorphic to  $Sh(h\mathcal{C}, h\mathcal{K})$  as defined in Section 1. This justifies the name of shape category for  $\mathcal{S}(\mathcal{C}, \mathcal{K})$  and of shape functor for  $(\Delta^* \circ \Gamma_E)_0 = S : \mathcal{C} \rightarrow \mathcal{S}(\mathcal{C}, \mathcal{K})$ .

**Definition 2.7.** The g.e. category  $\Sigma(\mathcal{C}, \mathcal{K})$  of the g.e. pair  $(\mathcal{C}, \mathcal{K})$  is the full image of the g.e. functor  $h_{\mathcal{GPD}}^* \circ \Gamma_E$ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Gamma_E} & \llbracket \mathcal{K}, \mathcal{GPD} \rrbracket^0 \xrightarrow{h_{\mathcal{GPD}}^*} \llbracket \mathcal{K}, h\mathcal{GPD} \rrbracket^0 \\ & \searrow \sigma=(h_{\mathcal{GPD}}^* \circ \Gamma_E)_0 & \nearrow (h_{\mathcal{GPD}}^* \circ \Gamma_E)_1 \\ & \Sigma(\mathcal{C}, \mathcal{K}) & \end{array}$$

Also  $\Sigma(\mathcal{C}, \mathcal{K})$  is locally discrete. It has the same objects as  $\mathcal{C}$  while morphisms are described as follows: if  $t \in \Sigma(\mathcal{C}, \mathcal{K})(X, Y)$ , then  $t : \pi E(-)^Y \rightarrow \pi E(-)^X$  is a natural transformation between the functors  $\pi E(-)^Y, \pi E(-)^X : \mathcal{K} \rightarrow h\mathcal{GPD}$ , hence it is a family of homotopy classes of functors of groupoids  $t_P : \pi P^Y \rightarrow \pi P^X$ , which is natural in  $P \in \mathcal{K}$ .

**Remark 2.8.** Since  $\Delta^* = h\Delta^* \circ h_{\mathcal{GPD}}^*$ , it follows from Lemma 1.1(i), that there exists a unique functor  $S : \Sigma(\mathcal{C}, \mathcal{K}) \rightarrow \mathcal{S}(\mathcal{C}, \mathcal{K})$  such that  $S = S \circ \sigma$ .

Moreover, since the homotopy functor is the identity on objects, from Lemma 1.1(ii) and Remark 2.5(4), it follows that  $\Sigma(\mathcal{C}, \mathcal{K})$  can be obtained also as the full image of the functor  $\Gamma_E^h$ , where  $E : \mathcal{K} \rightarrow \mathcal{C}$  is the inclusion.

### 3. Shape and strong shape equivalences

In the g.e. context described above strong shape equivalences have a nice categorical definition.

**Definition 3.1.** Let  $(\mathcal{C}, \mathcal{K})$  be a g.e. pair and let  $f : X \rightarrow Y$  be a 1-morphism of  $\mathcal{C}$ .

- (1)  $f$  is a shape equivalence (s.e., for short) for the pair  $(\mathcal{C}, \mathcal{K})$  if  $f_P^* : [Y, P] \rightarrow [X, P]$  is a bijection, for all  $P \in \mathcal{K}$ .
- (2)  $f$  is a strong shape equivalence (s.s.e., for short) for the pair  $(\mathcal{C}, \mathcal{K})$  if  $f_P^\# : \pi P^Y \rightarrow \pi P^X$  is an equivalence of groupoids, for all  $P \in \mathcal{K}$ .



Note that the shape functor  $\mathcal{S} : \mathcal{C} \rightarrow \mathcal{S}(\mathcal{C}, \mathcal{K})$  and the functor  $\sigma : \mathcal{C} \rightarrow \Sigma(\mathcal{C}, \mathcal{K})$  invert exactly the classes of shape equivalences and of strong shape equivalences for  $(\mathcal{C}, \mathcal{K})$ , respectively.

Recall that a continuous map  $f : X \rightarrow Y$  is called a (topological) s.e. if it satisfies the following two properties [20]:

- (s.e.1) for every  $g : X \rightarrow P$ ,  $P \in \text{ANR}$ , there is a map  $h : Y \rightarrow P$  such that  $h \circ f \simeq g$ ,
- (s.e.2) if  $h_0, h_1 : Y \rightarrow P$ ,  $P \in \text{ANR}$ , are such that  $h_0 \circ f \simeq h_1 \circ f$ , then  $h_0 \simeq h_1$ ,

while  $f$  is a (topological) s.s.e. whenever it satisfies (s.e.1) and the following strengthening of (s.e.2):

- (s.s.e.2) given  $h_0, h_1 : Y \rightarrow P$ ,  $P \in \text{ANR}$ , and a track  $\alpha : h_0 \circ f \Rightarrow h_1 \circ f$ , there is a unique track  $\beta : h_0 \Rightarrow h_1$  such that  $f^\#(\beta) = \alpha$ .

Condition (s.s.e.2) is stated in [20] in the following form: given  $h_0, h_1 : Y \rightarrow P$  and a homotopy  $F : h_0 \circ f \simeq h_1 \circ f$ , there is a homotopy  $G : h_0 \simeq h_1$  such that  $G \circ (f \times 1) \simeq F$  *rel end maps*. Since the homotopy  $G$  is then uniquely determined up to homotopies *rel end maps* [11, Proposition 1.2], it follows that a topological s.s.e. is just a strong shape equivalence for the pair  $(\mathcal{TOP}, \mathcal{ANR})$ , according to the previous definition.

Let us consider now the category  $\text{proTOP}$ , whose objects and morphisms we call *prospaces* and *promaps*, respectively.

The usual cylinder functor on  $\text{TOP}$  can be naturally extended to a cylinder functor on  $\text{proTOP}$ . For every *prospace*  $X = (X_a, x_{aa'}, A)$ , let  $X \times I = (X_a \times I, x_{aa'} \times 1, A)$ , where  $I$  is the unit interval. Two *promaps*  $f, g : X \rightarrow Y$  are globally homotopic if there is a homotopy  $H : X \times I \rightarrow Y$  such that  $H \circ e^0 = f$  and  $H \circ e^1 = g$ , where  $e^0, e^1 : X \rightarrow X \times I$  are the obvious level *promaps*. Consider also the existence of the level *promap*  $s : X \times I \rightarrow X$  such that  $s \circ e^\lambda = \text{identity}$ , for  $\lambda = 0, 1$ . Global homotopy is an equivalence relation on the class of morphisms of  $\text{proTOP}$  which is compatible with composition, hence one can consider the global homotopy classes, written  $[f]$ , that are the elements of the set  $[X, Y]$ . This last is the set of components of the track groupoid  $\pi Y^X$  of  $Y$  under  $X$ . If  $f, g : X \rightarrow Y$  are two *promaps* and  $H, H' : X \times I \rightarrow Y$  are two global homotopies connecting  $f$  and  $g$ , we say that they are global homotopic *rel end maps* if there is a higher homotopy  $\Phi : X \times I \times I \rightarrow Y$  satisfying the properties:

- (i)  $\Phi \circ e^0 = H$ ,
- (ii)  $\Phi \circ e^1 = H'$ ,
- (iii)  $\Phi \circ (e^0 \times 1) = f \circ s$ ,
- (iv)  $\Phi \circ (e^1 \times 1) = g \circ s$ .

Homotopy *rel end maps* is an equivalence relation and the class of  $H$ , denoted  $\{H\}$ , is a *track* from  $f$  to  $g$  (cf. [5]).

The notion of (topological) strong shape equivalence can be extended to *promaps* in the following way [11,20]:  $f : X \rightarrow Y$  is a s.s.e. in  $\text{proTOP}$  whenever the following two conditions hold:

- (pro s.s.e. 1) for every promap  $g: X \rightarrow P$ ,  $P \in \text{ANR}$ , there is a promap  $h: Y \rightarrow P$  such that  $h \circ f \simeq g$ ,
- (pro s.s.e. 2) given promaps  $g, h: Y \rightarrow P$ ,  $P \in \text{ANR}$ , and a track  $\alpha: g \circ f \Rightarrow h \circ f$ , there is a unique track  $\beta: g \Rightarrow h$  such that  $f^\#(\beta) = \alpha$ .

**Proposition 3.2.** *A promap  $f: X \rightarrow Y$  is a s.s.e. equivalence iff it is such for the g.e. pair  $(\text{proTOP}, \text{ANR})$ .*

In what follows we comment on the relationship of the Lisica–Mardešić strong shape category  $\text{SSH}(\text{TOP})$  of topological spaces with the category  $\Sigma(\text{TOP}, \text{ANR})$  as defined in Definition 2.7. In order to do this we need to recall some results contained in the paper [7] by Cathey and Segal. A key result there is the existence of a reflector (Proposition 1.10)  $\Psi: h(\text{proTOP}) \rightarrow h(\text{proTOP})_f$ , where  $h(\text{proTOP})_f$  is the full subcategory of fibrant prospaces, which can be restricted to a reflector  $\Psi: h(\text{proANR}) \rightarrow h(\text{proANR})_f$  (Proposition 2.3). The Steenrod homotopy category of prospaces  $ho(\text{proTOP})$  is defined to be the full image of the functor  $\Psi$ , that is

$$ho(\text{proTOP})(X, Y) \cong [\Psi(X), \Psi(Y)].$$

In [19] S. Mardešić has introduced the notion of resolution for a topological space and has proved that every space  $X$  has a canonically associated ANR-resolution  $r_X: X \rightarrow \widehat{X}$ , which is a morphism in  $\text{proTOP}$ . Notice that  $r_X$  is an ANR-resolution for  $X$  iff it is a s.s.e. for  $(\text{proTOP}, \text{ANR})$  [20]. Although Mardešić's correspondence is not functorial [24], however it gives a functor  $R: ho(\text{proTOP}) \rightarrow ho(\text{proANR})$ , which is reflective in the sense that

$$ho(\text{proTOP})(X, Y) \cong ho(\text{proANR})(\widehat{X}, Y),$$

for every  $X \in \text{TOP}$  and  $Y \in \text{proANR}$ .

The Lisica–Mardešić strong shape category  $\text{SSH}(\text{TOP})$  of topological spaces is obtained as the full image of the functor  $R$ , while the strong shape functor  $\text{SSH}: \text{TOP} \rightarrow \text{SSH}(\text{TOP})$  is just  $R_0$ .

**Proposition 3.3.**

- (i) *The following diagram is commutative (up to isomorphisms)*

$$\begin{array}{ccc}
 h\text{TOP} & & \\
 \downarrow R & \searrow \Gamma_E^h & \\
 ho(\text{proANR}) & & \llbracket \text{ANR}, h\text{GPD} \rrbracket^0 \\
 \downarrow \Psi_1 & \nearrow \Gamma_\alpha^h & \\
 h(\text{proANR})_f & & 
 \end{array}$$

where  $E: \text{ANR} \rightarrow \text{TOP}$ ,  $\alpha: \text{ANR} \rightarrow \text{proANR}$  are the embedding functors and  $\Gamma_\alpha^h$  is restricted to the full subcategory of  $h(\text{proANR})$  of fibrant prospaces.

- (ii) *There is a unique functor  $T: \text{SSH}(\text{TOP}) \rightarrow \Sigma(\text{TOP}, \text{ANR})$  such that  $T \circ \text{SSH} = \sigma$  and  $(\Gamma_E^h)_0 \circ T = \Gamma_\alpha^h \circ (\Psi_1 \circ R)_1$ . In particular  $T$  is the identity on objects.*

**Proof.** (i) For every  $X \in \text{TOP}$ , its ANR-resolution is a s.s.e.  $r_X: X \rightarrow \widehat{X} = \Psi_1(R(X))$ . Then, for all  $P \in \text{ANR}$ , there is an equivalence of groupoids  $(r_X)_P^\# : \pi P^X \simeq \pi P^{\widehat{X}}$ , hence an isomorphism in  $h\text{GPD}$ . It follows that  $\Gamma_E^h(X) \cong (\Gamma_\alpha^h \circ \Psi_1 \circ R)(X)$ .

(ii) By Remark 2.8 the category  $\Sigma(\mathcal{TOP}, \mathcal{ANR})$  can be also obtained as the full image of  $\Gamma_E^h$  while  $Ssh(\text{TOP})$  is the full image of  $\Psi_1 \circ R$ , since  $\Psi_1$  is fully faithful. By Lemma 1.1(i), there is a unique functor  $T: Ssh(\text{TOP}) \rightarrow \Sigma(\mathcal{TOP}, \mathcal{ANR})$ , such that  $T \circ Ssh = \sigma$  and  $(\Gamma_E^h)_0 \circ T = \Gamma_\alpha^h \circ (\Psi_1 \circ R)_1$ .  $\square$

Notice that the two categories  $\Sigma(\mathcal{TOP}, \mathcal{ANR})$  and  $Ssh(\text{TOP})$  have the same objects, namely all topological spaces. Moreover, a continuous map  $f: X \rightarrow Y$  is a s.s.e. exactly when  $\sigma(f)$  is an isomorphism in  $\Sigma(\mathcal{TOP}, \mathcal{ANR})$ .

A morphism  $\phi: X \rightarrow Y$  in  $\Sigma(\mathcal{TOP}, \mathcal{ANR})$  is a family of homotopy classes of functors of track groupoids  $\phi_P: \pi P^Y \rightarrow \pi P^X$ , which is natural in  $P \in \text{ANR}$ .

**Remarks 3.4.** The following facts were pointed out by the referee.

- (1) The functor  $T: Ssh(\text{TOP}) \rightarrow \Sigma(\mathcal{TOP}, \mathcal{ANR})$  above is clearly not full. It is neither faithful. Let  $Sol$  denote the dyadic solenoid. There exist two points in  $Sol$  such that the corresponding inclusions as one point spaces  $\{*\}$  into  $Sol$  give different strong shape morphisms, since they induce different morphisms in the Steenrod–Sitnikov homology [1]. However, they obviously induce homotopic functors of track groupoids  $\pi P^{Sol} \rightarrow \pi P^{\{*\}}$ , for all  $P \in \text{ANR}$ , hence they become equal in  $\Sigma(\mathcal{TOP}, \mathcal{ANR})$ .
- (2) The construction of the category  $\Sigma(\mathcal{C}, \mathcal{K})$  can be derived from a truncated version of Batanin’s bicategory of simplicial distributors [3], without considering coherence conditions between the isomorphisms involved in the definition of transformations of g.e. functors. First, one considers the bicategory whose objects are the g.e. categories, the 1-morphisms from  $\mathcal{C}$  to  $\mathcal{D}$  are g.e. functors of the form  $\mathcal{D}^0 \times \mathcal{C} \rightarrow \mathcal{GPD}$  and the 2-morphisms are homotopy classes of “incoherent” natural transformations between them. An incoherent natural transformation  $t$  between two g.e. functors  $F, G: \mathcal{C} \rightarrow \mathcal{GPD}$  is a family of functors  $t = \{t_X: F(X) \rightarrow G(X) \mid X \in \mathcal{C}\}$ , together with natural isomorphisms  $\psi_f: G(f) \circ t_X \cong t_Y \circ F(f)$ , for every 1-morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$ . Following the techniques from [1], one can show that this is indeed a bicategory, the only delicate point being the composition of 1-morphisms. This has both right and left extensions along 1-morphisms. Whenever  $E: \mathcal{K} \rightarrow \mathcal{C}$  is an embedding of g.e. categories, then there is a 1-morphism  $\phi$  from  $\mathcal{K}$  to  $\mathcal{C}$  given by  $\phi(X, Y) = \mathcal{C}(X, E(Y))$ . The right extension of  $\phi$  along itself will be a monad  $\mathcal{M}$  on  $\mathcal{C}$  such that  $\mathcal{M}(X, Y) = \text{Nat}_{\mathcal{K}}(\mathcal{C}(Y, E(-)), \mathcal{C}(X, E(-)))$ , where  $\text{Nat}_{\mathcal{K}}$  means homotopy classes of incoherent natural transformations on  $\mathcal{K}$ . The category  $\Sigma(\mathcal{C}, \mathcal{K})$  turns out to be the Kleisli category of this monad.

## Acknowledgements

The author wishes to thank Armin Frei, whose criticism was very helpful during the preparation of this paper.

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